Monte Carlo PDE Solver for Nonlinear Radiative Boundary Conditions

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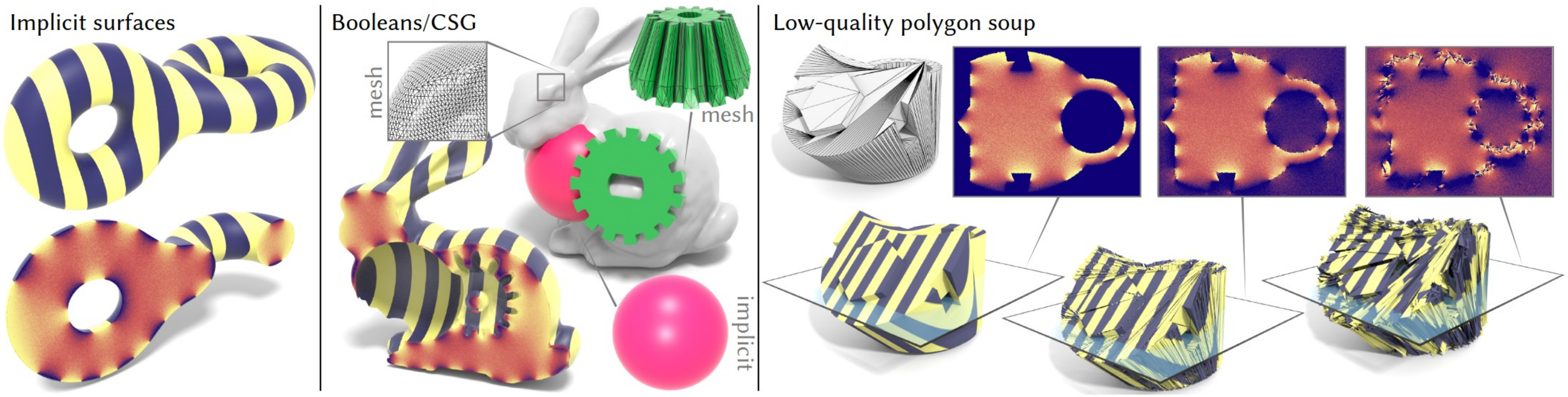
Monte Carlo Geometry Processing

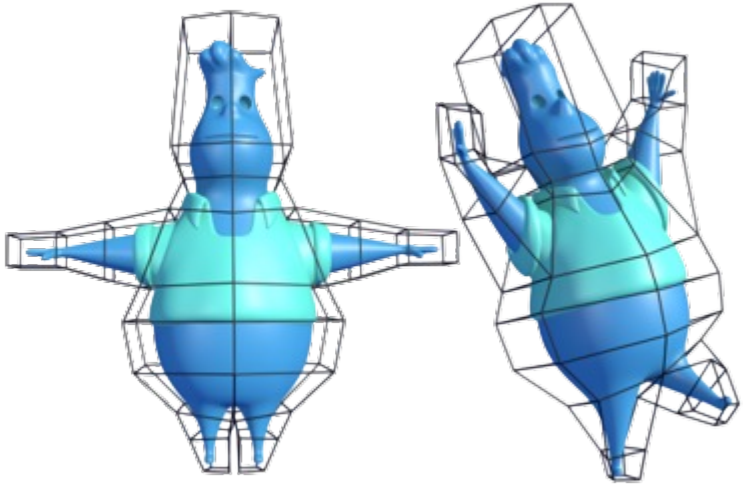
Monte Carlo Geometry Processing [Sawhney and Crane, 2020] provides

- A stochastic style solver for elliptic PDEs
- Works on imperfect geometry
- Allows different shape representations
- Natural parallelization



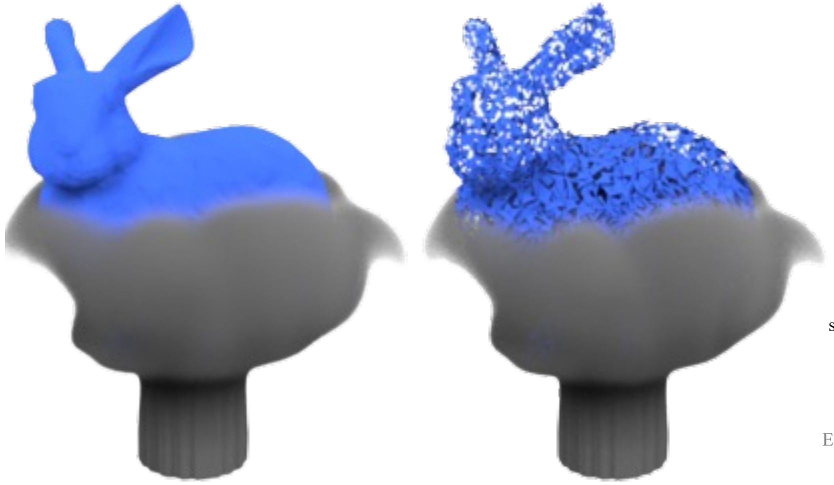
Monte Carlo PDE solvers are developing rapidly

Monte Carlo PDE Solvers are attractive due to their **impressive geometric flexibility**.



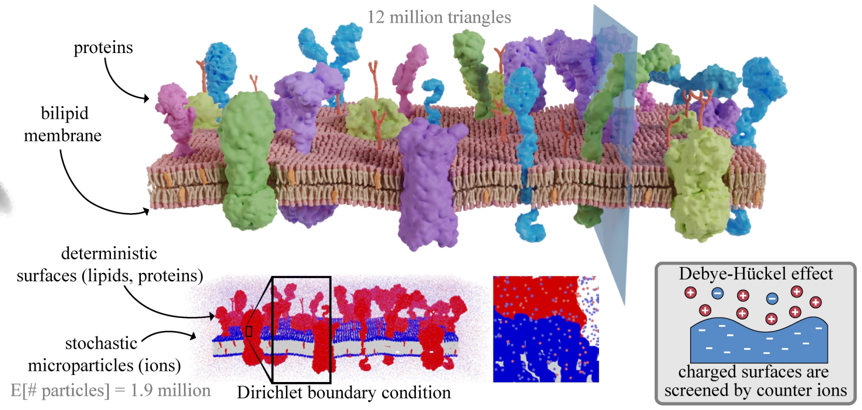
Shape deformation

de Goes and Desbrun 2024



Fluid simulation

Rioux-Lavoie et al. 2022



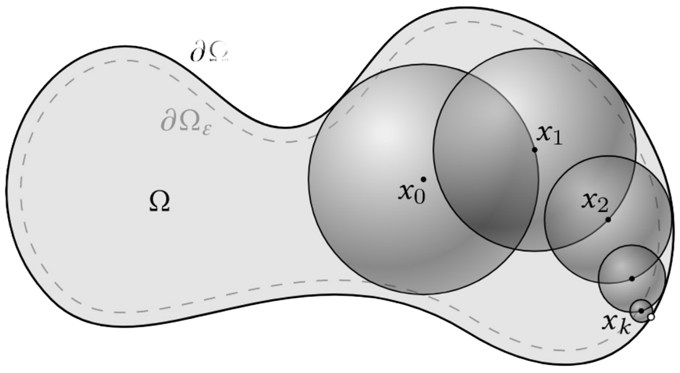
Participating media

Miller et al. 2025

A variety of applications in computer graphics

Monte Carlo PDE Solvers: Boundary Conditions

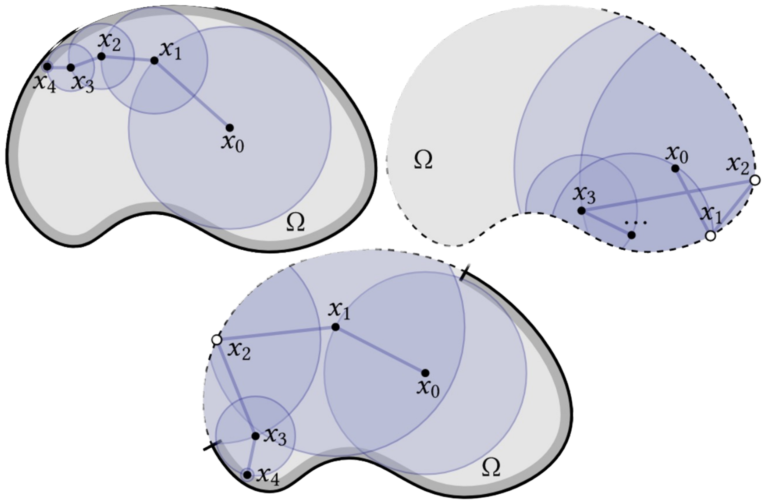
Existing Monte Carlo PDE Solvers can handle linear boundary conditions



Dirichlet boundary

Sawhney and Crane 2020

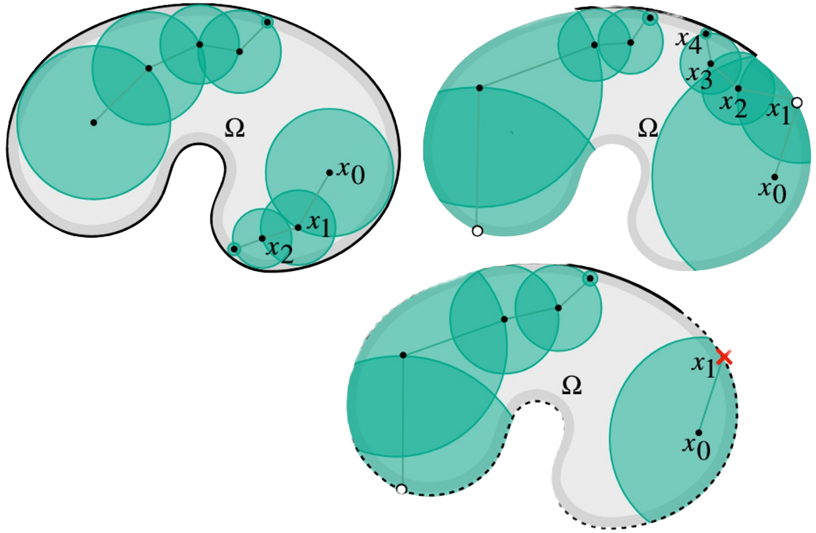
Fixed boundary temperature



Neumann boundary

Sawhney et al. 2023

Fixed boundary flux



Robin boundary

Miller et al. 2024

Convection

However, the linear boundary condition is not enough for many phenomena

Nonlinear boundary conditions: Heat transfer

The following boundary value problem governs the steady-state heat transfer

$$k_0 \Delta T(x) = 0$$ **conduction**

$$T(x) = T_0(x)$$ **fixed-temperature boundary**

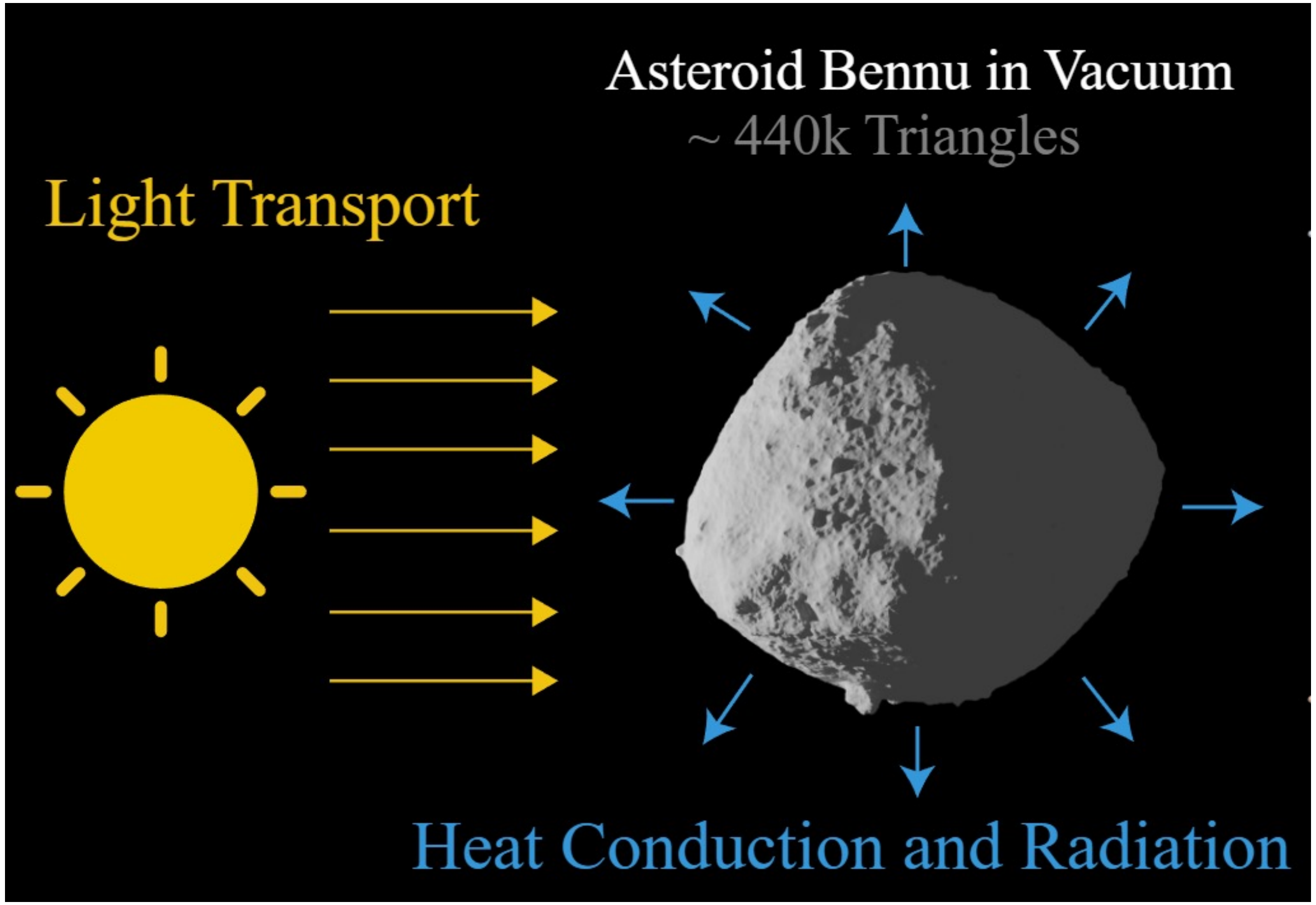
$$\frac{\partial T(x)}{\partial \vec{n}} = q_0(x)$$ **fixed-flux boundary**

$$\frac{\partial T(x)}{\partial \vec{n}} + \frac{h}{k_0}[T(x) - T_f(x)] + \frac{\sigma \epsilon}{k_0}[T(x)^4 - T_s(x)^4] = q_0(x)$$

convection **radiation**

Nonlinear boundary conditions: Application

Black/gray bodies are heated by radiation. What is the temperature distribution?



Methodology

Nonlinear boundary conditions: Approximation

To handle the nonlinear radiative boundary condition, we need an approximation

$$\frac{\partial u}{\partial \vec{n}}(x) + \mu(x)\, u^4(x) = h(x)$$

Nonlinear boundary condition

$$u^4 \approx u_0^3 u,$$

Linearization via cubic frozen

$$\frac{\partial u}{\partial \vec{n}}(x) + \mu(x) u_0^3(x) u(x) = h(x)$$

Robin condition

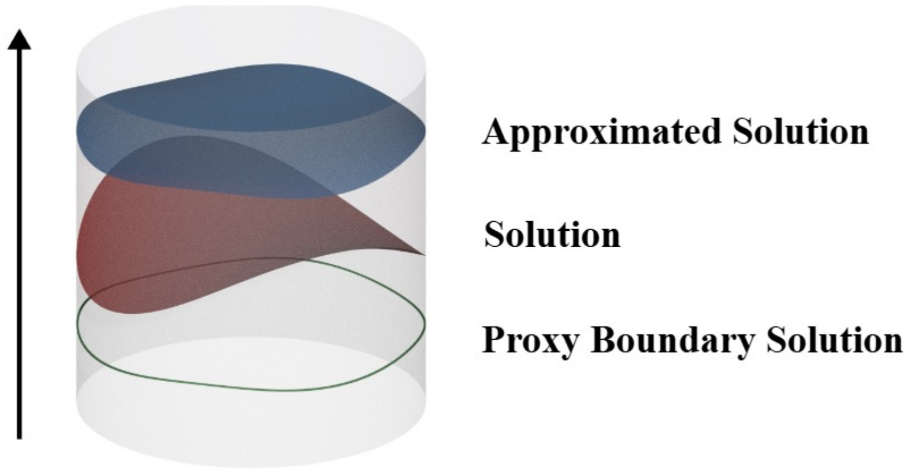
A Comparison Theorem: Approximation Error

If the initial guess is smaller than the true solution, the result will be larger.

LEMMA 4.1 (COMPARISON THEOREM). *Let u solve the nonlinear radiative boundary value problem* (11), *and let u_0 solve the linearized problem* (13) *with proxy $p(x) \geq 0$. If*

$$p(x) \leq u(x) \quad \text{for all } x \in \partial\Omega_R, \tag{14}$$

$$u_0(x) \geq u(x) \quad \text{for all } x \in \Omega. \tag{15}$$



Picard-style Fixed-Point Iteration

Consider the following boundary value problem

$$\begin{cases} u(x) = g(x), & x \in \partial\Omega_D, \\ \dfrac{\partial u}{\partial \vec{n}}(x) + \mu(x)\, u^4(x) = h(x), & x \in \partial\Omega_R, \\ \Delta u(x) = -f(x), & x \in \Omega. \end{cases}$$

Linearize the nonlinear part

$$\frac{\partial u}{\partial \vec{n}}(x) + \mu(x) u_0^3(x) u(x) = h(x)$$

Picard-style Fixed-Point Iteration

Define the **solution operator**, mapping the proxy function to the solution

$$\mathcal{S}(u_p, f, g, h)$$

Then, the solution of the original problem is the **fixed-point** of this operator

$$u = \mathcal{S}(u, f, g, h).$$

We solve the system with **Picard iteration**:

$$u_{n+1} = \mathcal{S}(u_n, f, g, h),$$

Picard-style Fixed-Point Iteration: Implementation

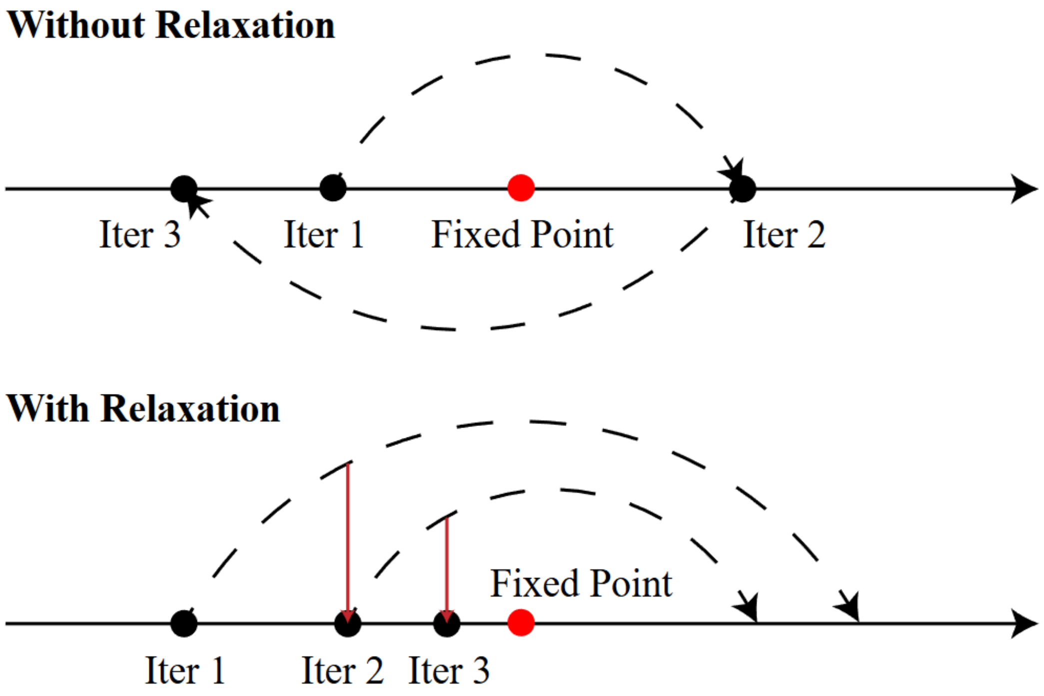
Branching estimation: Proxy Caching

$$\alpha(x)u_{n+1}(x) = \int_{\partial \mathrm{St}(x,R)} \rho_\mu(x,z)P^B(x,z)u_{n+1}(z)\,\mathrm{d}z + \int_{\partial \mathrm{St}_R(x,R)} G^B(x,z)h(z)\,\mathrm{d}z + \int_{\mathrm{St}(x,R)} G^B(x,y)f(y)\,\mathrm{d}y,$$

where the reflectance term ρ_μ is given by

$$\rho_\mu(x,z) = \begin{cases} 1 - \dfrac{G^B(x,z)}{P^B(x,z)}\mu(z)u_n^3(z), & z \in \partial \mathrm{St}_R, \\ 1, & z \in \partial \mathrm{St}_B. \end{cases}$$

Iteration may oscillate: relaxation

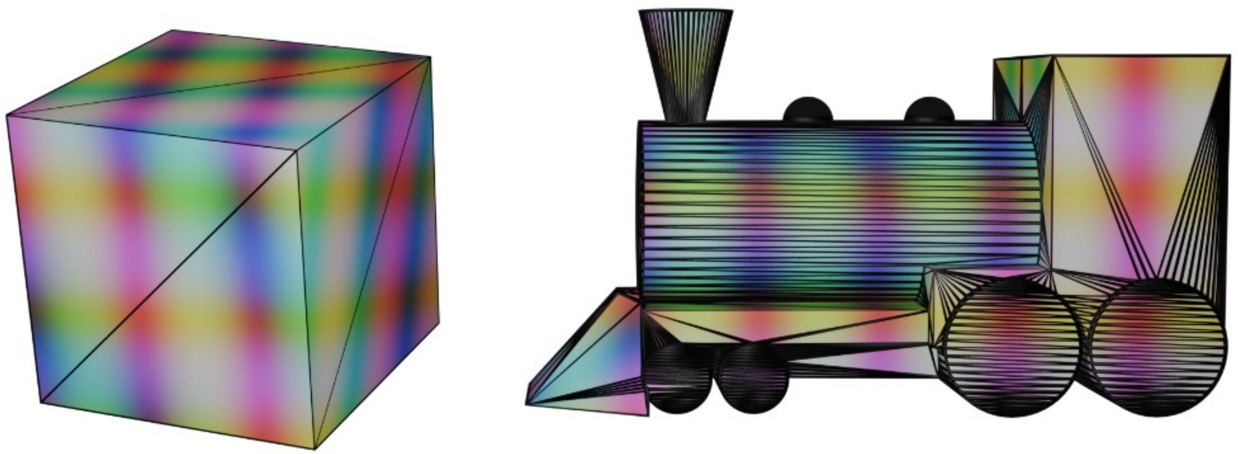


$$u_n \leftarrow \alpha u_n + (1-\alpha)u_{n-1}, \quad \alpha \in (0,1).$$

Proxy Function Representation: MLS approximation

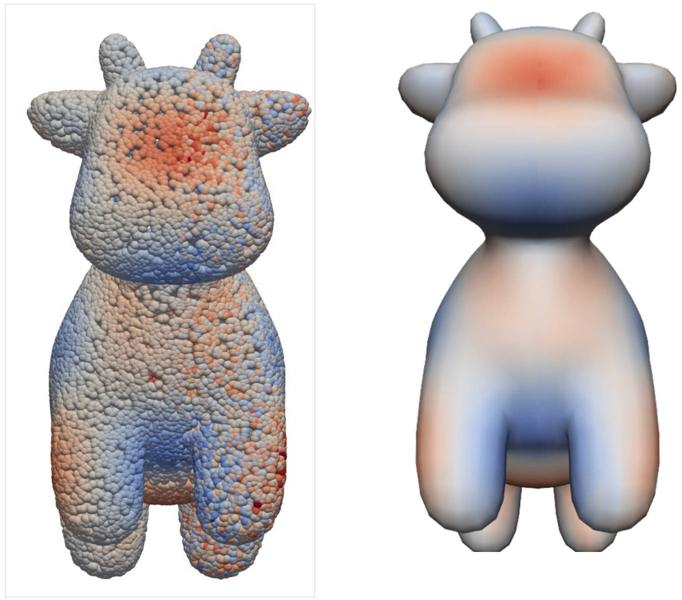
Per-vertex function is not a good choice

- Depend on the mesh scale
- Ignoring the high-frequency part



Point cloud + MLS approximation

- Point cloud with value
- MLS evaluation



$$E(a,b) = \sum_i w(\|x - P_i\|) \left(a + b^\top (P_i - x) - u_n(P_i)\right)^2$$

Picard-style Fixed-Point Iteration

Setup: LHS is the ground truth, and the exact solution is symmetric

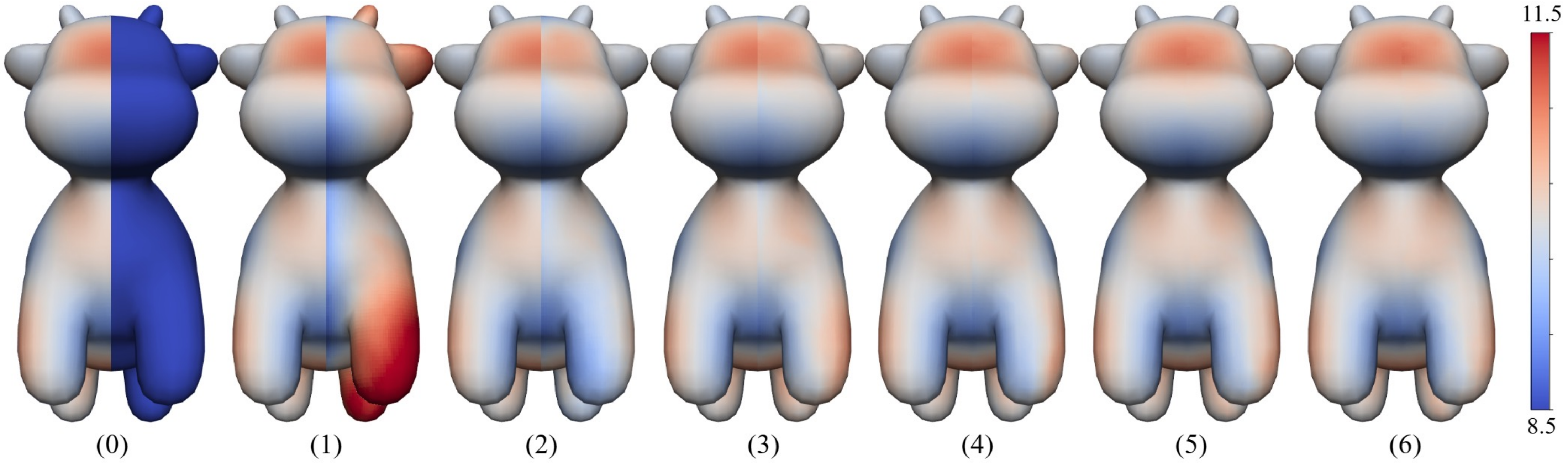
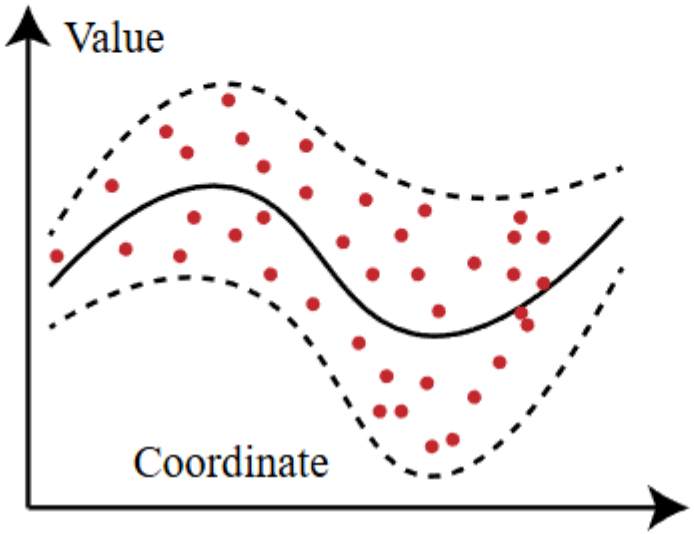


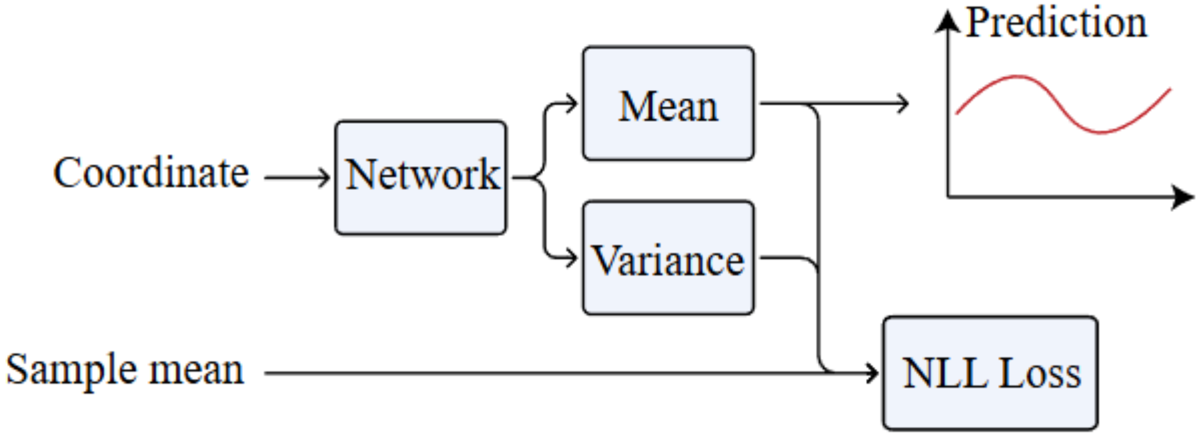
Fig. 2. Visualization of the fixed-point iteration process for the radiative boundary condition. The left half of each image shows the solution on the Dirichlet boundary, while the right half shows the corresponding solution on the radiative boundary. The ground-truth solution is symmetric across the two halves. Panel (0) shows the initial proxy, and panels (1)–(6) show the boundary solution after the n-th fixed-point iteration, illustrating progressive convergence toward the ground truth.

Denoising: Heteroscedastic Regression

We use a heteroscedastic regression to reduce Monte Carlo noise



(a) **Heteroscedastic Data**



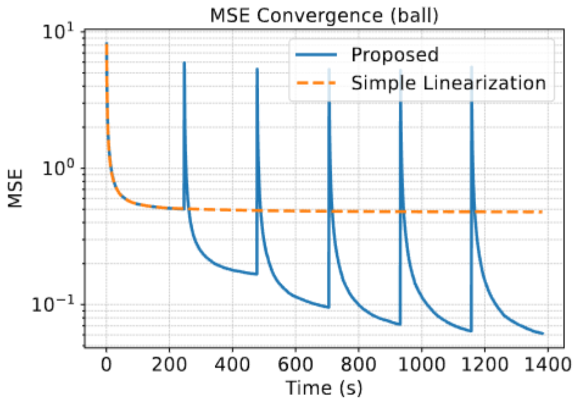
(b) **Regression**

Negative Log Loss: $$\mathcal{L}(\hat{u}, \sigma) = \mathbb{E}_{x \sim \partial\Omega}\left[\frac{(\hat{u}(x) - u(x))^2}{2\sigma^2(x)} + \frac{1}{2}\log \sigma^2(x)\right]$$

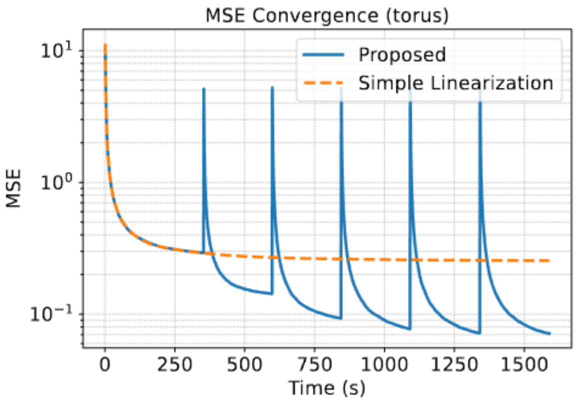
Evaluation

Evaluation with Synthetic Functions

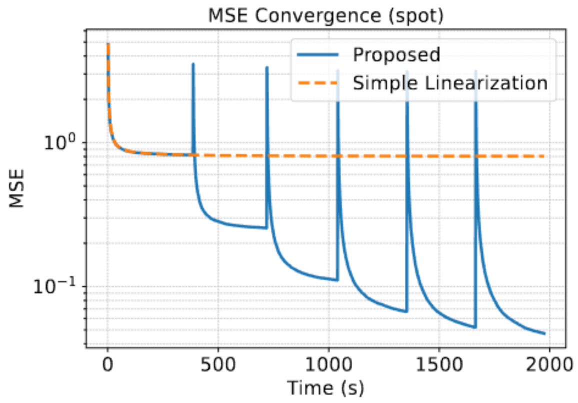
We evaluate our method and the linearization baseline with synthetic functions



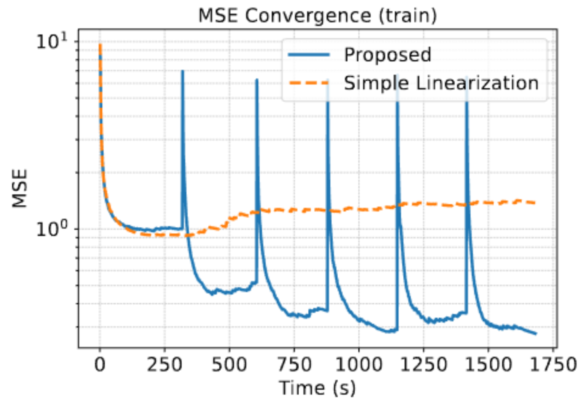
(a)



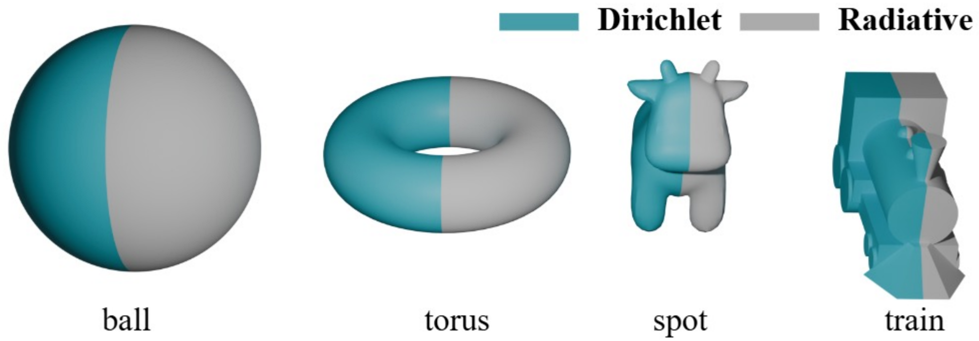
(b)



(c)



(d)



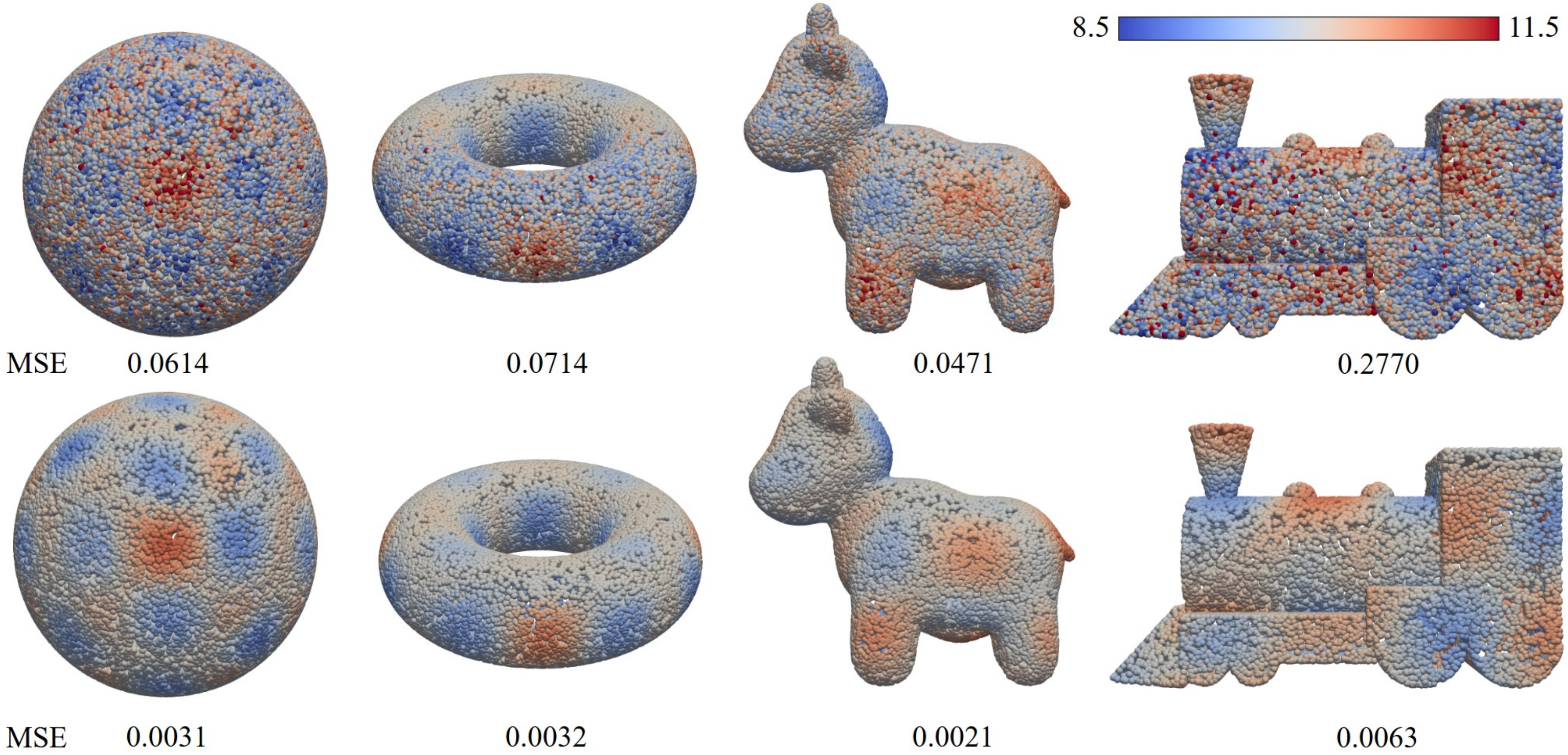
$$u(x) = 10 + \cos(\omega x)\cos(\omega y)\cos(\omega z),$$

$$\frac{\partial u}{\partial \vec{n}} + \gamma u^4 = f(x),$$

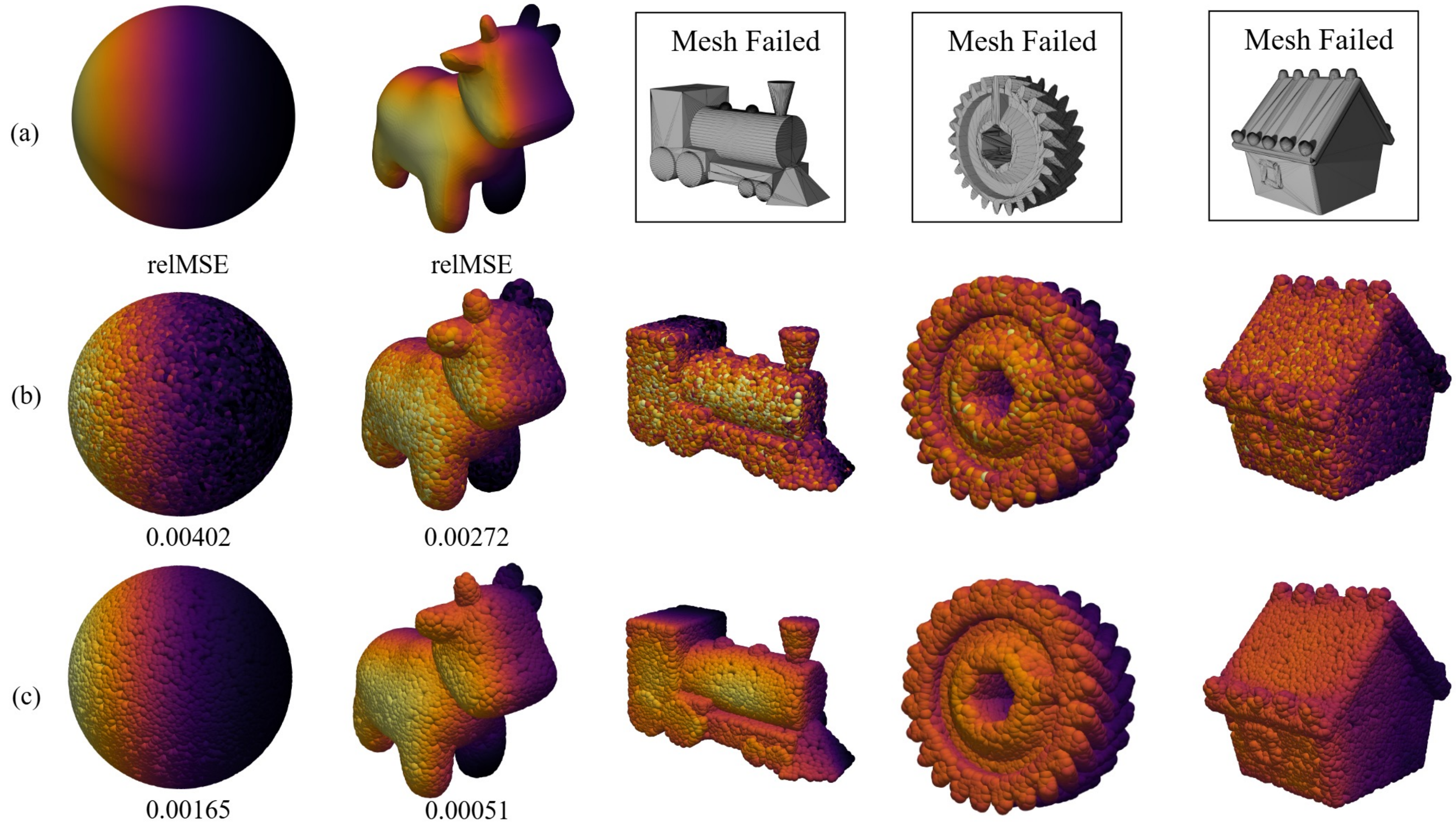
Denoising Results

W/o Denoising

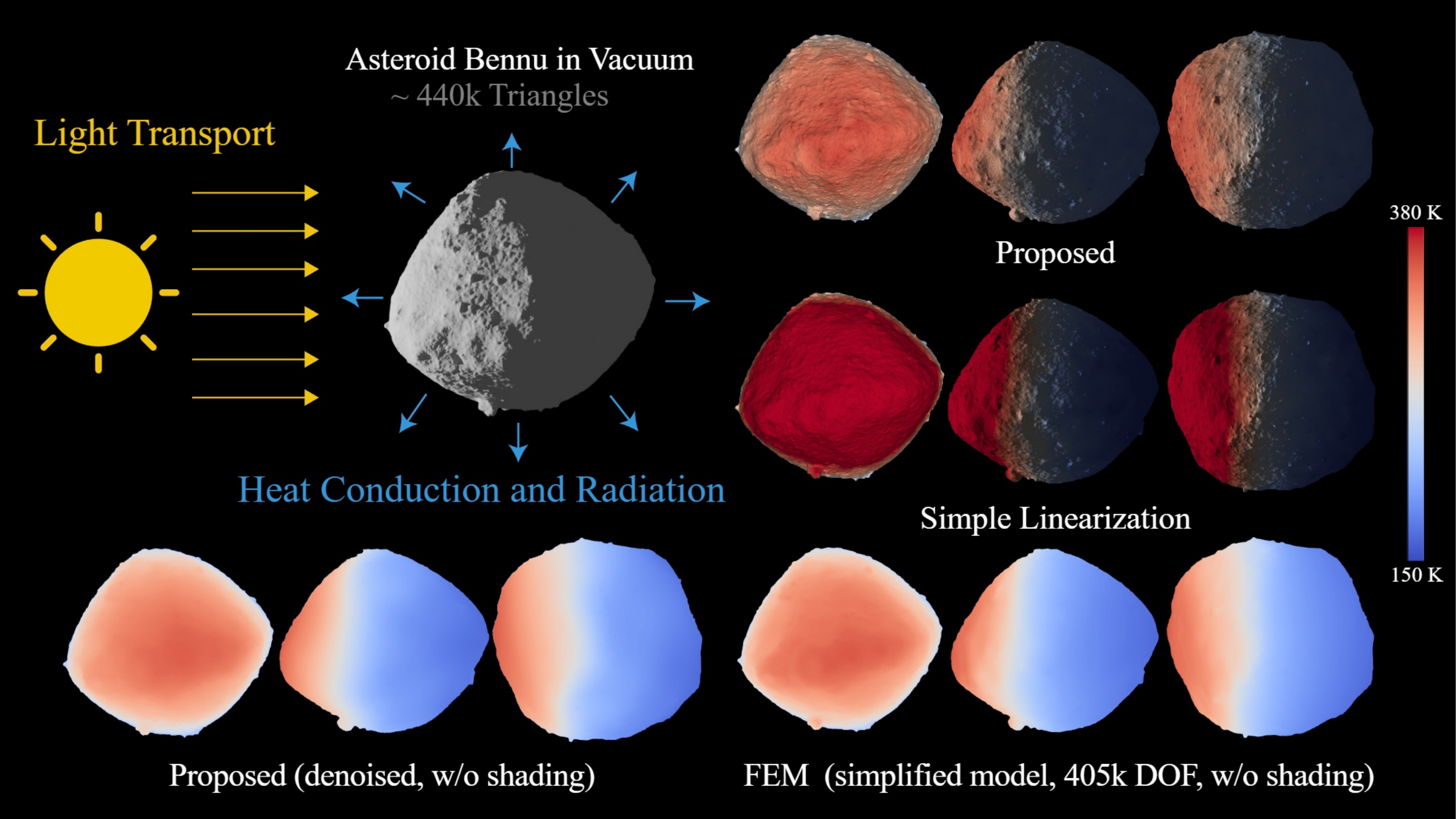
W/ Denoising



Light-Heat Coupled Problems: Radiation in Vacuum

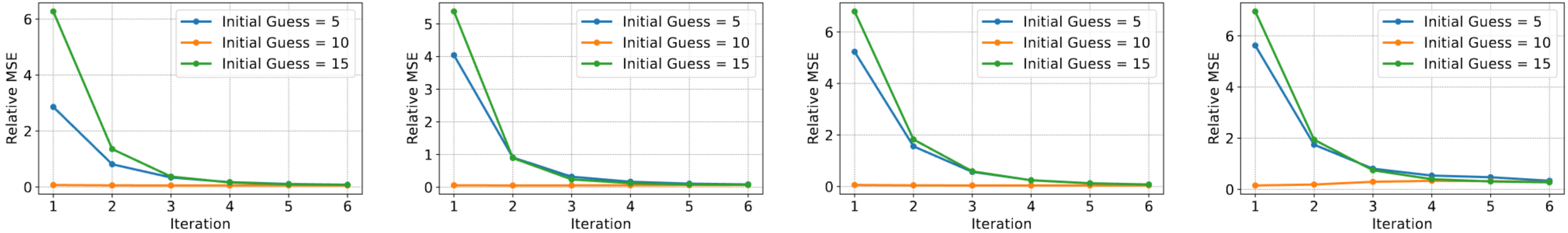


Comparison against FEM: MC converges to the same result and is more robust



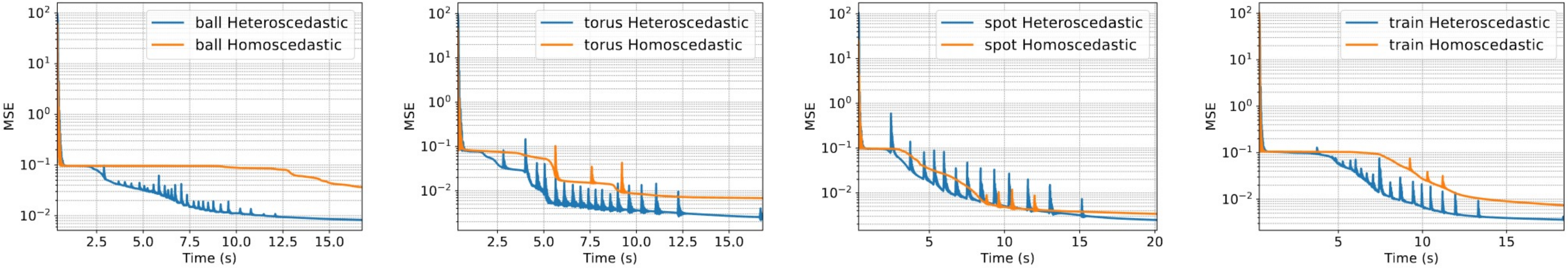
Asteroid Bennu in Vacuum
~ 440k Triangles
Light Transport
Heat Conduction and Radiation
Proposed
Simple Linearization
380 K
150 K
Proposed (denoised, w/o shading)
FEM (simplified model, 405k DOF, w/o shading)

Ablation Study and Effect of parameters



(a) (b) (c) (d)

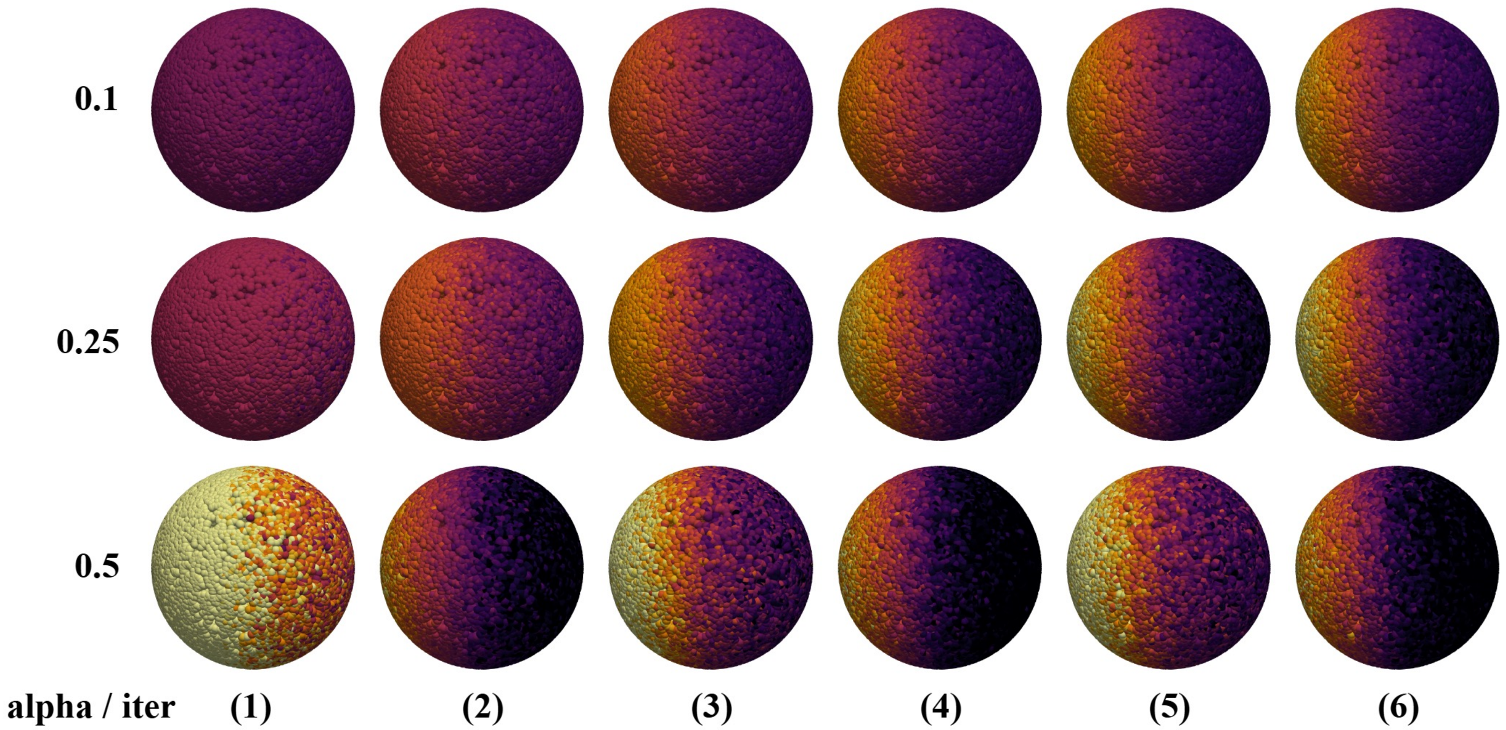
Different initial values

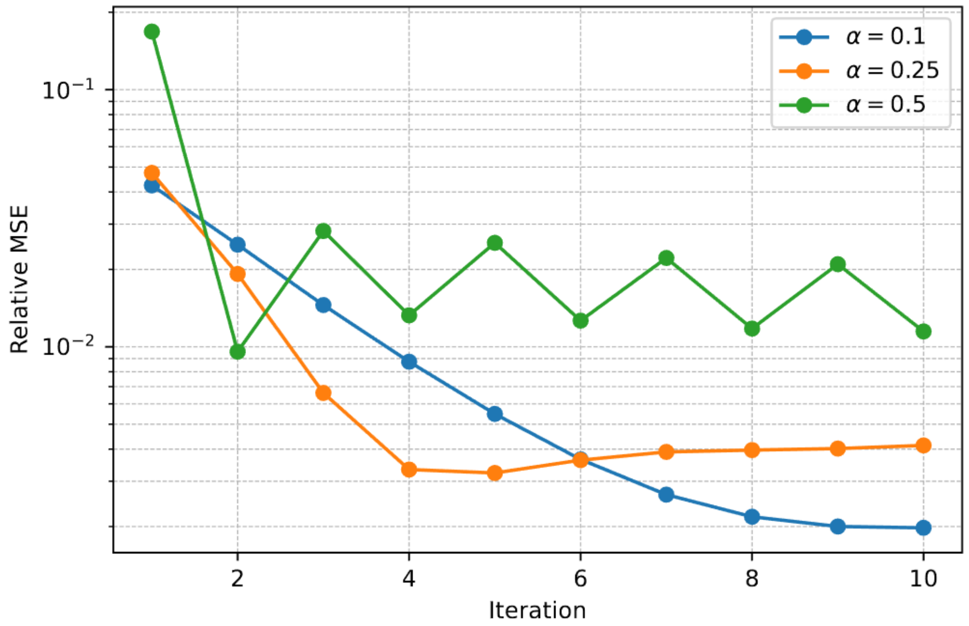


(a) (b) (c) (d)

Heteroscedastic Regression vs. Homoscedastic Regression.

Ablation Study and Effect of Parameters





Large relaxation may lead to oscillation, and small relaxation is safer

Summary

We proposed a Monte Carlo PDE solver for nonlinear radiative boundary conditions.
- Use Picard iteration instead of simple linearization
- Boundary function representation with MLS approximation
- A heteroscedastic regression denoising approach.

Advantages of the proposed method
- Compared with naïve linearization, the proposed method reduces systematic error
- Proposed denoising strategy can be applied to boundary functions
- Compared with FEM software, our approach is geometrically robust

Limitations and future work
- Acceleration: combine with previous variance reduction methods; Anderson acceleration and variants
- Transient problems for WoSt-type solvers
- Physical model: the emissivity may depend on temperature, material properties, or wavelength

Thanks